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A Note on the Algebraic Matrix Riccati Equation*

by

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Abstract

The results in [1,2] for the matrix Lyapunov equation are extended to the case of an algebraic matrix Riccati equation. Some errors in [1,2] are pointed out by a counter example. The estimations obtained in this note are shown to be exact for certain cases. Similar results are possible for the discrete algebraic matrix Riccati equation.

See Form 1473

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I. Introduction

Lower bounds of the solution K to the matrix Lyapunov equation

$$A'K + KA + Q = 0 \tag{1}$$

have been obtained in [1,2] in terms of $\{A,Q\}$. Equation (1) is related to the linear constant homogeneous system

$$\dot{x}(t) = Ax(t), \quad x(t_o) = x_o. \tag{2}$$

The well known result is that the system (2) is asymptotically stable if and only if for each positive definite matrix Q there exists a positive definite solution K to (1). We consider in this note the algebraic matrix Riccati equation

$$A'K + KA - KBB'K + Q = 0 \tag{3}$$

which is related to the linear constant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_o) = x_o. \tag{4}$$

The corresponding result is that the system (4) is stabilizable if and only if for each positive definite matrix Q there exists a positive definite solution K to (3). We will obtain lower bounds of the solution to (3) in this note. Results for (3) should coincide with those of (1) when $B = 0$. Bounds for the finite time solution of the matrix differential Riccati equation are given in [4,5] for continuous systems, and in [6,7] for discrete systems, in terms of the controllability and observability matrices. These are compared with the results in this note for some special cases. Applications of bounds for (1) are listed in [1,2] and similar applications for bounds of (3) are possible.

II. Main Results

In the following, $|x|$ denotes the Euclidean vector norm and $\|C\|$ the matrix norm induced by the Euclidean norm, i.e.,

$$\|C\| = \sup_{|x|=1} |Cx| = \max_i \lambda_i^{1/2}(C'C) \tag{5}$$

where $\lambda_i(M)$ denotes an eigenvalue of a matrix M. For a nonnegative definite matrix S it holds that $\|S\| = \sup_{|x|=1} x'Sx = \lambda_{max}(S)$ and $\inf_{|x|=1} x'Sx = \lambda_{min}(S)$.

Theorem 1. Assume that Q is a symmetric positive definite matrix. The positive definite solution matrix K to the algebraic matrix equation (3) has the following bounds:

$$K \geq \frac{\lambda_{min}(Q)}{\|A\| + \{\|A\|^2 + \|BB'\| \lambda_{min}(Q)\}^{1/2}} I \tag{6}$$

$$\|K\| \geq \frac{\lambda_{max}(Q)}{\|A\| + \{\|A\|^2 + \|BB'\| \lambda_{max}(Q)\}^{1/2}} . \tag{7}$$

Proof: Pre- and post-multiplying (3) by x' and x respectively yields

$$x'KBB'Kx - 2x'A'Kx - x'Qx = 0,$$

which can be expressed as

$$x'KBB'Kx + 2|x'A'Kx| - x'Qx \geq 0. \tag{8}$$

From the Schwarz Inequality we have

$$|x'A'Kx| \leq |Ax||Kx| \tag{9}$$

$$x'KBB'Kx \leq |BB'Kx||Kx| \leq \|BB'\| \, |Kx|^2. \tag{10}$$

Combining (8), (9), and (10) yields

$$\|BB'\| \cdot |Kx|^2 + 2|Ax| \cdot |Kx| - x'Qx \geq 0. \tag{11}$$

Since $|Kx|$ is nonnegative, it should satisfy

$$|Kx| \geq \frac{1}{\|BB'\|} \{- |Ax| + \{|Ax|^2 + \|BB'\| x'Qx\}^{1/2}\} = \frac{x'Qx}{|Ax| + \{|Ax|^2 + \|BB'\| x'Qx\}^{1/2}} . \tag{12}$$

The last equality in (12) follows from the following identity

$$\{- a + \sqrt{a^2 + bc}\} \{a + \sqrt{a^2 + bc}\} = bc.$$

Dividing both sides of (12) by $|x|$ yields

$$\frac{|Kx|}{|x|} \geq \frac{\dfrac{x'Qx}{|x|^2}}{\dfrac{|Ax|}{|x|} + \left\{\dfrac{|Ax|^2}{|x|^2} + \| BB' \| \dfrac{x'Qx}{|x|^2}\right\}^{1/2}} \geq \frac{\dfrac{x'Qx}{|x|^2}}{\| A\| + \left\{\| A\|^2 + \| BB' \| \dfrac{x'Qx}{|x|^2}\right\}^{1/2}} \, . \tag{13}$$

For $a > 0$ and $b > 0$, it can be shown that the function

$$f(x) = \frac{x}{a + \sqrt{a^2 + bx}} \tag{14}$$

is monotonically increasing with respect to x for $x \geq 0$. Thus we have

$$K \geq \lambda_{min}(K)I = \inf_x \frac{|Kx|}{|x|} I \geq \frac{\lambda_{min}(Q)}{\| A\| + \{\| A\|^2 + \| BB' \| \lambda_{min}(Q)\}^{1/2}} I \tag{15}$$

and

$$\| K \| = \lambda_{max}(K) \geq \frac{\lambda_{max}(Q)}{\| A\| + \{\| A\|^2 + \| BB' \| \lambda_{max}(Q)\}^{1/2}} \, . \tag{16}$$

This completes the proof.

By taking $B = 0$, the results in (6) and (7) coincide with those of [1,2]. The results in (6) and (7) are sharp in the sense that its estimations are exact in some cases as shown in the following example. It is also demonstrated in the example that the estimations in Theorem 1 are better than those of [4,8] for some special cases of $Q > 0$.

Example: (a) Let $A = -I$, $B = I$, $Q = 3I$. Then $\| A\| = \| BB' \| = 1$. From (6) and (7), $K \geq I$ and $\| K \| \geq 1$. The exact solution to (3) is $K = I$. This shows that the estimations in (6) and (7) are exact in this case. In [4] it is suggested that $K \geq \left(\dfrac{2}{3(1-e^{-2T})} + \dfrac{e^{2T}-1}{2}\right)^{-1}$ for any $T > 0$, which implies $K \geq \frac{3}{4}(\sqrt{3} - 1)I$. Also in [8] it is suggested that $K \geq \frac{3}{4} I$.

(b) Let $A = 0$, $B = bI$, and $Q = q^2 I$. From (6) and (7) $K \geq \frac{q}{b} I$ and $\| K \| \geq \frac{q}{b}$. The exact solution to (3) is $K = \frac{q}{b} I$. Thus the estimations of (6) and (7) in this case are exact. It is suggested in [4] that $K \geq (\frac{1}{q^2 T} + b^2 T)^{-1} I$ for

any $T \geq 0$, which implies $K \geq \frac{q}{2b} I$.

By a slight modification the results in Theorem 1 can be applied to the discrete algebraic matrix Riccati equation

$$K = \Phi' K \Phi - \Phi' K B (I + B' K B)^{-1} B' K \Phi + Q, \tag{17}$$

which can be transformed to

$$K \Phi^{-1} B B' K + K \Phi^{-1} - (Q \Phi^{-1} B B' + \Phi') K - Q \Phi^{-1} = 0. \tag{18}$$

The corresponding result can be given as follows.

Theorem 2. Assume that Φ is nonsingular, Q nonnegative definite, and $(Q\Phi^{-1} + \Phi'^{-1} Q)$ nonsingular. The positive definite solution matrix K to the discrete algebraic matrix Riccati equation (17) has the following bounds:

$$K \geq 2 \frac{\gamma_{min}(Q\Phi^{-1})}{\|G\| + \{\|G\|^2 + 4\|BB'\Phi'^{-1}\| \gamma_{min}(Q\Phi^{-1})\}^{1/2}} I \tag{19}$$

$$\|K\| \geq 2 \frac{\gamma_{max}(Q\Phi^{-1})}{\|G\| + \{\|G\|^2 + 4\|BB'\Phi'^{-1}\| \gamma_{max}(Q\Phi^{-1})\}^{1/2}} \tag{20}$$

where

$$G = \Phi^{-1} - \Phi - BB'\Phi'^{-1} Q \tag{21}$$

$$\gamma_{min}(C) = \frac{1}{2} \min_i |\lambda_i(C + C')| \tag{22}$$

$$\gamma_{max}(C) = \frac{1}{2} \max_i |\lambda_i(C + C')| \tag{23}$$

The results in Theorem 2 can be obtained from the fact that $\inf_{|x|=1} |x'Cx| = \gamma_{min}(C)$ and $\sup_{|x|=1} |x'Cx| = \gamma_{max}(C)$. By taking $B = 0$ the results in Theorem 2 apply to the discrete Lyapunov matrix equation, $\Phi' K \Phi - K = -Q$. It is also true that the estimations in (19) and (20) are exact for some special cases and thus better than those estimations in [7] in certain cases.

III. Correction to [2,1]

In this section we use the same notations as [2,1]. Since the equation (17)

in [2],

$$\|(A')^{-1}PA^{-1}\|_2 = \frac{\beta_n}{\sigma_1}, \tag{24}$$

is not generally true, Theorem 2 in [2] which claims

$$\alpha_n \geq \frac{\beta_n}{2\sigma_1^{1/2}} \tag{25}$$

is invalid.

Counter example: Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$. Then $A'^{-1}PA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\| A'^{-1}PA^{-1}\|_2 = 1$, $\beta_n = 2$, and $\sigma_1 = 1$. Thus the inequality (24) does not hold. The solution matrix Q of the Lyapunov equation is $Q = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Thus $\alpha_n = \frac{1}{2}$ which shows that the relation (25) is invalid.

The relation (25) also appeared in [1] and should be changed to $\alpha_n \geq \frac{\beta_n}{2\sigma_n^{1/2}}$, which can be obtained from (7) by taking $B = 0$.

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